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A characterization of even doubly-stochastic matrices

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Abstract

Even doubly-stochastic matrices are characterized with the aid of the minima of functionals defined by the even diagonals contained in the matrix.

Keywords: Doubly-stochastic matrices; Even doubly-stochastic matrices; Even and odd diagonals

1. Introduction and preliminaries

In 1961, L. Mirsky gave the following necessary condition for a doubly-stochastic $n \times n$-matrix with $n \geq 3$ to be even, i.e. a convex combination of even permutation matrices,

$$\forall \pi \in A_n,\ \forall k \in \{1, \ldots, n\} : \sum_{i=1}^{n} a_{i\pi(i)} - a_{k\pi(k)} \leq n - 3, \tag{1}$$

and he conjectured that this condition would also be sufficient [10]. This is true for $n = 3$, but for $n \geq 4$, this is false, see [1], and a characterization of the polyhedron defined by these doubly-stochastic matrices remained desirable. In 1991 R.A. Brualdi and B.L. Liu [4] conjectured that the number of linear inequalities necessary to characterize the even permutation matrices is not bounded by a polynomial in n. This conjecture was confirmed by W. H. Cunningham and Y. Wang [5] and J. Hood and D. Perkinson [6] in 2004, see also [3,4] for related topics. In the present paper we present a characterization of even doubly-stochastic matrices in terms of the even diagonals contained in the matrix, of the number of fixed points of their quotients and of a natural quadratic functional associated to them. It yields criteria to decide whether a given doubly-stochastic matrix is even or not. Moreover, it can be detected whether the convex combination in terms of permutation matrices is unique or not. The main result of Section 2 can be summarized as follows: For a given doubly-stochastic matrix $A = \left(a_{ij}\right)_{n\times n}$, denote its distinct even diagonals by $\pi_1, \ldots, \pi_p$ and set $b = (b_m)_{p\times 1}$ with

$$b_m = \sum_{i=1}^{n} a_{i\pi_m(i)},$$

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and introduce the matrix $H = 2\left(\mathrm{tr}(\pi_m \pi_q^{-1})\right)_{p\times p}$. Then H is semi-positive definite, see Lemma 2.2, and A is even iff the equation $Hy = 2b$ has a non-negative solution y with coordinate sum equal to 1 such that $f(y) = 0$ where

$$f(x) = \sum_{i,j=1}^{n} \left(a_{ij} - \sum_{k=1}^{p} x_k \delta_{\pi_k(i)j} \right)^2 .$$

In that case, the convex representation is unique if $\det H > 0$, while there can be several distinct ones if $\det H = 0$. In Section 3 the characterization is refined in the presence of an even diagonal, while Section 4 presents some examples.

By definition, a doubly-stochastic matrix is a real non-negative square matrix whose sum of each row and of each column amounts to 1. The polyhedron of all doubly-stochastic $n \times n$ matrices will be denoted by Ω_n. A famous result by G. Birkhoff states that the permutation matrices are precisely the vertices of Ω_n, see e.g. [11]. The symmetric group on n elements will be denoted by S_n and the alternating group by A_n. We say that a permutation $\sigma \in S_n$ is *contained* in a real matrix $A = \left(a_{ij}\right)_{n\times n}$ or a *diagonal* of A if

$$\prod_{i=1}^{n} a_{i\sigma(i)} \neq 0, \tag{2}$$

which will be denoted by $\sigma \prec A$. Unless precision is required, we shall identify a permutation $\pi \in S_n$ with the corresponding permutation matrix

$$\pi = P_\pi = (\delta_{\pi(i)j})_{n\times n}.$$

The polyhedron of even doubly-stochastic matrices, i.e. the convex hull of the alternating group A_n in Ω_n will be denoted by Δ_n. Accordingly, a convex combination of odd permutation matrices will be called an *odd* doubly-stochastic matrix. This apparent parity is neither exclusive nor complete: There are doubly-stochastic matrices that are even and odd, while there are others that are neither even nor odd. Conceivably, we call a matrix A *totally even* (respectively *totally odd*) if no odd permutation (resp. no even permutation) is contained in A. In 1916, Denes König [7], see also [8], showed that any doubly-stochastic matrix possesses a diagonal and, thereby, is a convex combination of permutation matrices using Birkhoff's algorithm, see e.g. [8]. In [2] the existence of even diagonals and their minimal number in a given doubly-stochastic matrix as a function of the number of positive elements has been established.

Throughout we shall use the following notations.

Definition 1.1.

$\mathcal{M}_n(\mathbb{R})$ = algebra of real $n \times n$ matrices
$\mathbf{I} = \mathbf{I}_n = n \times n$-identity matrix
$\mathbf{e} = \mathbf{e}_n = n \times 1$-column vector with constant entries equal to 1
$\mathrm{tr}(A)$ = trace of the matrix A
$\mathbb{E}^p = \left\{x \in (\mathbb{R}^+)^p \mid x^t \mathbf{e}_n = 1\right\}$
$\mathbb{F}^p = \{x \in (\mathbb{R}^+)^p \mid x^t \mathbf{e}_n \leq 1\}$.

For further matrix theoretical terminology we refer to [9].

The algebra $\mathcal{M}_n(\mathbb{R})$ is endowed with the Euclidean or *Frobenius*-norm

$$\|A\| := \sqrt{\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}^2} = \sqrt{\mathrm{tr}(AA^t)}.$$

This norm is not an operator norm, but it is consistent with the Euclidean norm in $\mathbb{R}^n$ and fulfills $\|AB\| \leq \|A\| \|B\|$.

2. The variational approach

For a given matrix $A \in \Omega_n$, let $\pi_1, \ldots, \pi_p$ denote the distinct even diagonals of A, $\{\pi \in A_n \mid \pi \prec A\}$ and define $f = f_A : \mathbb{R}^p \longrightarrow \mathbb{R}^+$ by

$$f(x_1, \ldots, x_p) = \left\| A - \sum_{k=1}^{p} x_k \pi_k \right\|^2 = \sum_{i,j=1}^{n} \left(a_{ij} - \sum_{k=1}^{p} x_k \delta_{\pi_k(i)j} \right)^2 . \tag{3}$$

We note in passing that $p = \frac{1}{2}\left(\mathrm{per}(P(A)) + \det(P(A))\right)$, where $P(A)$ is the adjacency matrix defined by A. For more details and further estimates related to p we refer to [2]. Since $f(x) \to +\infty$ as $\|x\| \to +\infty$, f attains its infimum in $\mathbb{R}^p$ and, thereby, always has critical points. Clearly, if $A \in \Omega_n$ is even, then there is some non-negative vector $y \in \mathbb{R}^p$ with $y^t \mathbf{e} = 1$ such that $f(y) = 0$ and

$$A \in \Delta_n \Rightarrow \min_{x \in \mathbb{R}^p} f(x) = 0. \tag{4}$$

But this condition is by no means sufficient, see Example 4.3 and it has to be complemented by an extra condition in order to characterize Δ_n. Therefore, we have to analyse in detail the critical points of the functional f. Denote the *Hessian matrix* of f by

$$H(f)(x_1, \ldots, x_p) = \left(\frac{\partial^2 f}{\partial x_i \partial x_j} \right)_{p \times p} (x_1, \ldots, x_p).$$

We readily calculate

$$\begin{aligned} \frac{\partial f}{\partial x_l}(x_1, \ldots, x_p) &= \sum_{i=1}^{n} \sum_{j=1}^{n} 2(-\delta_{j\pi_l(i)}) \left(a_{ij} - \sum_{k=1}^{p} x_k \delta_{j\pi_k(i)} \right) \\ &= -2 \sum_{i=1}^{n} \left(a_{i\pi_l(i)} - \sum_{k=1}^{p} x_k \delta_{\pi_l(i)\pi_k(i)} \right). \end{aligned}$$

Thus

$$\frac{1}{2} \frac{\partial^2 f}{\partial x_q \partial x_l}(x_1, \ldots, x_p) = \sum_{i=1}^{n} \delta_{\pi_l(i)\pi_q(i)}.$$

Since the r.h.s. is just the number of common images of π_l and π_q, i.e. the number of fixed points of $\pi_l \pi_q^{-1}$, $H(f)$ is seen to be a constant symmetric non-negative matrix given by

Lemma 2.1. $H(f) = 2 \left(\mathrm{tr}(\pi_m \pi_q^{-1}) \right)_{p \times p}$.

Especially, $H(f)$ does not depend on the critical point $\nabla f(x) = 0$. This fact enables us to show the definite character of $H(f)$.

Lemma 2.2. *The matrix $H(f)$ is semi-positive definite.*

Proof. Introduce the following even doubly-stochastic matrix

$$B = \sum_{k=1}^{p} \frac{1}{p} \pi_k.$$

Then B contains exactly the same even diagonals as A and

$$H(f_A) = 2 \left(\mathrm{tr}\left(\pi_l \pi_q^{-1} \right) \right)_{q,l} = H(f_B).$$

But f_B attains its minimum 0 by Eq. (4), which shows that $H(f_B)$ is semi-positive definite, as well as $H(f_A)$. □

Thus $H(f)$ cannot have negative eigenvalues. If all the eigenvalues are positive, then A is not necessarily even, as the following example shows.

Example 2.3. For $n = 3$ choose $A = \frac{1}{4}(123) + \frac{1}{4}(132) + \frac{1}{4}1_{S_n} + \frac{1}{4}(12) = \frac{1}{4}\begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix} \notin \Delta_3$. The even diagonals are $\pi_1 = (123)$, $\pi_2 = (132)$, $\pi_3 = 1_{S_n}$ and $H(f) = 2\begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ having the determinant 216.

An element $y = (y_1, \ldots, y_p) \in \mathbb{R}^p$ is a critical point of f iff for each $l = 1, \ldots, p$

$$\begin{aligned} &\frac{\partial f}{\partial x_l}(y_1, \ldots, y_p) = 0 \\ &\iff \sum_{i=1}^{n}\left(a_{i\pi_l(i)} - \sum_{k=1}^{p} y_k \delta_{\pi_l(i)\pi_k(i)}\right) = 0 \\ &\iff \sum_{i=1}^{n} a_{i\pi_l(i)} = \sum_{k=1}^{p} y_k \sum_{i=1}^{n} \delta_{\pi_l(i)\pi_k(i)}. \end{aligned}$$

This corresponds to the p equations

$$\begin{cases} \displaystyle\sum_{k=1}^{p} \operatorname{tr}(\pi_1\pi_k^{-1}) y_k = \sum_{i=1}^{n} a_{i\pi_1(i)} \\ \qquad\vdots \\ \displaystyle\sum_{k=1}^{p} \operatorname{tr}(\pi_p\pi_k^{-1}) y_k = \sum_{i=1}^{n} a_{i\pi_p(i)} \end{cases}$$

or

$$M\begin{pmatrix} y_1 \\ \vdots \\ y_p \end{pmatrix} = b$$

with

$$M = \begin{pmatrix} \operatorname{tr}(\pi_1\pi_1^{-1}) & \cdots & \operatorname{tr}(\pi_1\pi_p^{-1}) \\ \vdots & & \vdots \\ \operatorname{tr}(\pi_p\pi_1^{-1}) & \cdots & \operatorname{tr}(\pi_p\pi_p^{-1}) \end{pmatrix} = \frac{1}{2}H(f) \quad \text{and} \quad b = \left(\sum_{i=1}^{n} a_{i\pi_l(i)}\right)_{p\times 1}.$$

Thus we can state

Lemma 2.4. *A vector $y \in \mathbb{R}^p$ is a critical point of the functional f iff*

$$\frac{1}{2}H(f)y = \left(\operatorname{tr}(\pi_l\pi_q^{-1})\right)_{p\times p} y = \left(\sum_{i=1}^{n} a_{i\pi_l(i)}\right)_{p\times 1}. \tag{5}$$

If $\det H(f) > 0$ then (5) has a unique solution, namely

$$y = (y_1, \ldots, y_p)^t = 2H(f)^{-1}\begin{pmatrix} \displaystyle\sum_{i=1}^{n} a_{i\pi_1(i)} \\ \vdots \\ \displaystyle\sum_{i=1}^{n} a_{i\pi_p(i)} \end{pmatrix}, \tag{6}$$

and there is exactly one critical point. Since $H(f)$ is positive definite, we are led to

Corollary 2.5. *If* $\det H(f) > 0$, *then the doubly-stochastic matrix* A *is even iff the vector* y *defined in Eq.* (6) *is non-negative and satisfies* $y^t\,\mathbf{e} = 1$ *and* $f(y) = 0$.

Without the constraint $f(y) = 0$ the result is false. Take the Example 2.3, where the unique solution of $H(y)y = 2b$ is given by $y^t = \frac{1}{3}(1, 1, 1)$. Since $\det H(f) > 0$, y is the absolute minimum, but $f(y) > 0$ and $A \notin \Delta_3$.

The condition $\det H(f) = 0$ can be fulfilled by even and non-even matrices. Take e.g. $A = \frac{1}{2}(123) + \frac{1}{2}(456) \in \Delta_6$ having the even diagonals. $\pi_1 = (123)$, $\pi_2 = (456)$, $\pi_3 = (123)(456)$ and $\pi_4 = 1_{S_n}$ and

$$\det H(f_A) = \begin{vmatrix} 12 & 0 & 6 & 6 \\ 0 & 12 & 6 & 6 \\ 6 & 6 & 12 & 0 \\ 6 & 6 & 0 & 12 \end{vmatrix} = 0.$$

A non-even example is given by Example 4.3. For the general case, let us derive the following lemmata.

Lemma 2.6. *Suppose* $\det H(f) = 0$. *If* f *admits a minimum* 0, *then this minimum is not isolated and* $\dim_{\mathbb{R}}\langle \pi_k \mid 1 \le k \le p\rangle < p$.

Proof. Suppose $f(y) = 0$ with $y = (y_1, \ldots, y_p) \in \mathbb{R}^p$. Then $\nabla f(y) = 0$ and

$$\frac{1}{2}H(f)y = b,$$

where we have used Lemma 2.4. Since $q := \operatorname{rank} H(f) < p$, y is seen to belong to an affine subspace of dimension $p - q$ of critical points at which f vanishes.

Finally, the existence of distinct vectors $y = (y_1, \ldots, y_p)$ and $z = (z_1, \ldots, z_p)$ with $f(y) = f(z) = 0$ implies

$$A = \sum_{k=1}^{p} y_k\pi_k = \sum_{k=1}^{p} z_k\pi_k \quad \text{and} \quad \sum_{k=1}^{p}(y_k - z_k)\pi_k = 0.$$

This shows that the diagonals of A, $\pi_1, \ldots \pi_p$ are linearly dependent. $\square$

Lemma 2.7. *The functional* f *is constant on the set of critical points of* f *being precisely*

$$\mathcal{S} := \{y \in \mathbb{R}^p \mid H(f)y = 2\,b\}.$$

Proof. Indeed, since $\mathcal{S}$ is either a singleton or an affine subspace of $\mathbb{R}^p$, $\mathcal{S}$ is connected, and ∇f vanishes in $\mathcal{S}$. This permits us to conclude. $\square$

Lemma 2.8. *If* $\dim_{\mathbb{R}}\langle \pi_k \mid 1 \le k \le p\rangle < p$ *then* $\det H(f) = 0$.

Proof. Suppose that there are reals $\lambda_1, \ldots, \lambda_p$, not all vanishing, such that

$$\sum_{k=1}^{p} \lambda_k\pi_k = 0.$$

Then for each $i \in \{1 \ldots p\}$, the matrix $\sum_{k=1}^{p} \lambda_k\pi_k\pi_i^{-1}$ is the zero matrix and, thereby,

$$\sum_{k=1}^{p} \lambda_k \operatorname{tr}(\pi_k\pi_i^{-1}) = 0,$$

showing that the rows of $H(f)$ are linearly dependent. $\square$

Corollary 2.9. *Suppose* $A \in \Delta_n$. *Then the following conditions are equivalent:*

(a) $\det H(f) = 0$.
(b) $\dim_{\mathbb{R}}\langle \pi_k \mid 1 \le k \le p\rangle < p$.

(c) *The matrix A can be written in more than one way as a linear combination of even permutation matrices contained in A.*

Note that $\det H(f) = 0$ can hold, while there is only one way to express A as a convex combination of even permutation matrices, see Example 4.2. If $f(y) = 0$ for some $y \in \mathcal{S} \cap \mathbb{E}^p$, then the matrix A must be even. Thus we can state the following criteria.

Theorem 2.10. *A doubly-stochastic matrix A is even iff $\mathcal{S} \cap \mathbb{E}^p \cap f^{-1}(0) \neq \emptyset$.*

Corollary 2.11. *If $f(z) > 0$ for some $z \in \mathcal{S}$, then A is not even.*

Clearly the same holds if $f > 0$ in $\mathbb{E}^p$.

3. Taking into account one even diagonal

The above characterization can be improved by bearing in mind that an even doubly-stochastic matrix necessarily possesses even diagonals. It is not necessary to include the case of totally odd doubly-stochastic matrices. Thus, assuming that $A \in \Omega_n$ has the even diagonal $\sigma \in A_n$, all entries of the principal diagonal in the matrix $\sigma^{-1}A$ are positive. Working with the latter matrix instead of A, we can assume w.l.o.g. that

$$1_{S_n} \prec A. \tag{7}$$

Let $\pi_1, \ldots, \pi_r$ denote the distinct even diagonals of A different from 1_{S_n}, i.e. $r = p - 1$ and $\pi_p = 1_{S_n}$. The aim is to control whether the matrix A can be written in the form

$$\sum_{k=1}^{r} x_k \pi_k + \left(1 - \sum_{k=1}^{r} x_k\right) 1_{S_n}$$

with $(x_1, \ldots, x_r)$ varying in the polyhedron $\mathbb{F}^r = \left\{x \in (\mathbb{R}^+)^r \mid x_1 + \ldots + x_r \leq 1\right\}$ or not. Accordingly, we modify the functional f into $g = g_A : (\mathbb{R}^+)^r \longrightarrow \mathbb{R}^+$ by setting

$$\begin{aligned} g(x_1, \ldots, x_r) &= \left\| A - \sum_{k=1}^{r} x_k \pi_k - (1 - \sum_{k=1}^{r} x_k) 1_{S_n} \right\|^2 \\ &= \sum_{i,j=1}^{n} \left(a_{ij} - \sum_{k=1}^{p} x_k \delta_{\pi_k(i)j} - \left(1 - \sum_{k=1}^{p} x_k\right) \delta_{ij} \right)^2. \end{aligned}$$

The minimum of g restricted to $\mathbb{F}^r$ is always attained therein, but in general, these minima are not critical points of g. But, if the minimal value of g amounts to 0, then the minimum is a critical point and shows that A is even:

$$A \in \Delta_n \iff \min_{\mathbb{F}^p} g = 0. \tag{8}$$

Lemma 3.1. *The Hessian matrix of the functional g is given by*

$$H(g) = 2\left(n + \operatorname{tr}\left(\pi_m \pi_q^{-1} - \pi_m - \pi_q\right)\right)_{r \times r}.$$

It is a constant, symmetric, non-negative and semi-positive definite matrix.

Proof. For $1 \leq m, q \leq r$ we have

$$\begin{aligned} \frac{\partial g}{\partial x_m}(x_1, \ldots, x_r) &= 2 \sum_{i=1}^{n} \sum_{j=1}^{n} \left(-\delta_{j\pi_m(i)} + \delta_{ij}\right) \left(a_{ij} - \sum_{k=1}^{r} x_k \delta_{\pi_k(i)j} - \delta_{ij} + \sum_{k=1}^{r} x_k \delta_{ij} \right) \\ &= 2 \sum_{i=1}^{n} \left(a_{ii} - \sum_{k=1}^{r} x_k \delta_{i\pi_k(i)} - 1 + \sum_{k=1}^{r} x_k - a_{i\pi_m(i)} + \sum_{k=1}^{r} x_k \delta_{\pi_m(i)\pi_k(i)} + \delta_{i\pi_m(i)} - \sum_{k=1}^{r} x_k \delta_{\pi_m(i)i} \right). \end{aligned}$$

Thus

$$\begin{aligned}\frac{\partial^2 f}{\partial x_q \partial x_m}(x_1,\ldots,x_r) &= 2\sum_{i=1}^{n}\sum_{j=1}^{n}\left(-\delta_{j\pi_m(i)}+\delta_{ij}\right)\left(-\delta_{j\pi_q(i)}+\delta_{ij}\right)\\ &= 2\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\delta_{j\pi_m(i)}\delta_{j\pi_q(i)}+\delta_{ij}^2-\delta_{j\pi_m(i)}\delta_{ij}-\delta_{j\pi_q(i)}\delta_{ij}\right)\\ &= 2\sum_{i=1}^{n}\left(\delta_{\pi_m(i)\pi_q(i)}+1-\delta_{\pi_m(i)i}-\delta_{\pi_q(i)i}\right)\\ &= 2\sum_{i=1}^{n}\left(\delta_{\pi_m(i)\pi_q(i)}\right)+2n-2\sum_{i=1}^{n}\left(\delta_{\pi_m(i)i}\right)-2\sum_{i=1}^{n}\left(\delta_{\pi_q(i)i}\right)\\ &= 2n+2\,\mathrm{tr}\left(\pi_l\pi_q^{-1}-\pi_l-\pi_q\right).\end{aligned}$$

But $\mathrm{tr}\left(\pi_m\pi_q^{-1}-\pi_m-\pi_q\right)$ counts just the elements of the common support of π_m and π_q, reduced by the number of fixed points of π_m and reduced by the number of fixed points of π_q that are not fixed points of π_m. Since the three involved sets are mutually disjoint, $\mathrm{tr}\left(\pi_m\pi_q^{-1}-\pi_m-\pi_q\right)$ is bounded from below by $-n$. Thus $H(g)$ is non-negative.

Finally, for the semi-positive definite character, we observe as in the proof of Lemma 2.2 that for the even doubly-stochastic matrix

$$B=\sum_{k=1}^{r}\frac{1}{r+1}\pi_k+\frac{1}{r+1}1_{S_n}$$

the Hessian matrices of g coincide:

$$H(g_B)=2\left(n+\mathrm{tr}\left(\pi_l\pi_q^{-1}-\pi_l-\pi_q\right)\right)_{q,l}=H(g_A).$$

By Eq. (8), g_B attains its absolute minimum 0. Thus $H(g_B)$ is semi-positive definite as well as $H(g_A)$. $\square$

Introduce the vector $c=(c_m)_{r\times 1}$ with

$$c_m=n+\sum_{i=1}^{n}\left(a_{i\pi_m(i)}-a_{ii}-\delta_{i\pi_m(i)}\right).$$

Lemma 3.2. *A vector $y\in\mathbb{R}^r$ is a critical point of the functional g iff*

$$\frac{1}{2}H(g_A)\,y=\left(\sum_{k=1}^{r}y_k\left(n+\mathrm{tr}\left(\pi_m\pi_k^{-1}-\pi_k-\pi_m\right)\right)\right)_{r\times 1}=c. \tag{9}$$

Proof. For $1\le m\le r$, $\frac{\partial f}{\partial x_m}(x_1,\ldots,x_r)=0$ iff

$$\begin{aligned}\sum_{i=1}^{n}\left(1-a_{ii}+a_{i\pi_m(i)}-\delta_{i\pi_m(i)}\right) &= \sum_{i=1}^{n}\left(\sum_{k=1}^{r}x_k-\sum_{k=1}^{r}x_k\delta_{i\pi_k(i)}+\sum_{k=1}^{r}x_k\delta_{\pi_m(i)\pi_k(i)}-\sum_{k=1}^{r}x_k\delta_{\pi_m(i)i}\right)\\ &= \sum_{k=1}^{r}x_k\left(\sum_{i=1}^{n}\left(1-\delta_{i\pi_k(i)}+\delta_{\pi_m(i)\pi_k(i)}-\delta_{\pi_m(i)i}\right)\right).\end{aligned}$$

Thus, $y=(y_1,\ldots,y_r)^t$ is a critical point of g iff for all m

$$c_m=\sum_{k=1}^{r}y_k\left(n+\mathrm{tr}\left(\pi_m\pi_k^{-1}-\pi_k-\pi_m\right)\right)=\left(\frac{1}{2}H(g_A)y\right)_m. \quad\square$$

We note in passing that a critical point of g can yield the absolute minimum in $\mathbb{F}^p$ while being of positive value. Take e.g. the matrix

$$A = \frac{1}{4}\begin{pmatrix}1 & 2 & 1\\ 2 & 1 & 1\\ 1 & 1 & 2\end{pmatrix} \quad \text{with} \quad H(g) = 2\begin{pmatrix}6 & 3\\ 3 & 6\end{pmatrix} \quad \text{and} \quad c = \begin{pmatrix}3\\ 3\end{pmatrix}$$

having the non-trivial even diagonals $\pi_1 = (123)$ and $\pi_2 = (132)$. The unique solution of $H(g)y = 2c$ is given by $y^t = \left(\frac{1}{3}, \frac{1}{3}\right)$ and yields an absolute minimum $g(y) > 0$ in $\mathbb{F}^2$. Thus, we conclude that $A \notin \Delta_3$.

As above, the affine subspace of $\mathbb{R}^r$

$$\mathcal{R} := \{y \in \mathbb{R}^r \mid H(g)y = 2\,c\}$$

is connected, and the functional g defined on $\mathbb{R}^r$ is constant on the set of its critical points being precisely $\mathcal{R}$. Now we can state the following

Theorem 3.3. *Suppose $A \in \Omega_n$ with $1_{S_n} \prec A$.*

(a) *If* $\det H(g) > 0$, *then $A \in \Delta_n$ iff the vector $y = 2H(g)^{-1}c$ belongs to $\mathbb{F}^p \cap g^{-1}(0)$.*

(b) *If* $\det H(g) = 0$, *then $A \in \Delta_n$ iff $\mathcal{R} \cap \mathbb{F}^p \cap g^{-1}(0) \neq \emptyset$.*

Corollary 3.4. *If $g(z) > 0$ for some $z \in \mathcal{R}$, then A is not even.*

As pointed out above, the condition $1_{S_n} \prec A$ causes no restriction. If A has no even diagonal, then evidently $A \notin \Delta_n$. If A has the diagonal $\sigma \in A_n$, then $A \in \Delta_n$ iff $\sigma^{-1}A \in \Delta_n$.

4. Examples

Example 4.1. The (even) doubly-stochastic matrix

$$A = \frac{1}{10}(123) + \frac{3}{10}(134) + \frac{4}{10}(145) + \frac{2}{10}(12345) = \frac{1}{10}\begin{pmatrix}0 & 3 & 3 & 4 & 0\\ 0 & 7 & 3 & 0 & 0\\ 1 & 0 & 4 & 5 & 0\\ 3 & 0 & 0 & 1 & 6\\ 6 & 0 & 0 & 0 & 4\end{pmatrix}$$

has the non-trivial even diagonals $\pi_1 = (123)$, $\pi_2 = (134)$, $\pi_3 = (145)$ and $\pi_4 = (12345)$, and

$$M := \frac{1}{2}H(f) = \begin{pmatrix}5 & 1 & 0 & 2\\ 1 & 5 & 1 & 1\\ 0 & 1 & 5 & 2\\ 2 & 1 & 2 & 5\end{pmatrix} \quad \text{with} \quad \det M = 390 \quad \text{and} \quad b = \frac{1}{10}\begin{pmatrix}12\\ 22\\ 27\\ 23\end{pmatrix}.$$

The vector $y = M^{-1}b = \frac{1}{10}\begin{pmatrix}1\\ 3\\ 4\\ 2\end{pmatrix}$ yields exactly the original representation of A in Δ_n.

Example 4.2. Let $A \in \Delta_4$ be positive. Then the whole alternating group consists in diagonals of A that are numbered by $\pi_1 = 1_{S_n}$, $\pi_2 = (123)$, $\pi_3 = (132)$, $\pi_4 = (124)$, $\pi_5 = (142)$, $\pi_6 = (134)$, $\pi_7 = (143)$, $\pi_8 = (234)$, $\pi_9 = (243)$,

$\pi_{10} = (12)(34)$, $\pi_{11} = (13)(24)$, $\pi_{12} = (14)(23)$. Then

$$\frac{1}{2}H(f) = \begin{pmatrix} 4 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 4 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 4 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 4 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 4 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 4 & 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 4 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 4 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 4 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 4 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 4 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 4 \end{pmatrix} \quad \text{with rank } H(f) = 10.$$

Thus the affine subspace $\mathcal{S}$ has dimension 2. For

$$A = \frac{1}{6}\begin{pmatrix} 1 & 3 & 1 & 1 \\ 3 & 1 & 1 & 1 \\ 1 & 1 & 3 & 1 \\ 1 & 1 & 1 & 3 \end{pmatrix} \quad \text{with} \quad b = \frac{1}{6}(8, 8, 8, 8, 8, 4, 4, 4, 4, 8, 4, 4)^t$$

the elements of $\mathcal{S} = \{x \in \mathbb{R}^{12} \mid H(f)x = 2b\}$ have the form[1]

$$\left(t + \frac{1}{6}, \frac{1}{6} + s, \frac{1}{6} - s - t, \frac{1}{6} - s - t, \frac{1}{6} + s, s, -s - t, -s - t, s, t + \frac{1}{6}, t, t\right)$$

with $(s, t) \in \mathbb{R}^2$. Looking for those elements that are non-negative and have coordinate sum 1, we note that for $s = t = 0$ the vector

$$y := \left(\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, 0, 0, 0, 0, \frac{1}{6}, 0, 0\right)$$

fulfills the requirements and yields $A \in \Delta_4$ with

$$A = \frac{1}{6}(1_{S_n} + (123) + (132) + (124) + (142) + (12)(34)).$$

Example 4.3. With the matrix $H(f)$ from the preceding examples, for

$$B = \frac{1}{10}\begin{pmatrix} 1 & 7 & 1 & 1 \\ 7 & 1 & 1 & 1 \\ 1 & 1 & 7 & 1 \\ 1 & 1 & 1 & 7 \end{pmatrix} \quad \text{with} \quad b = \frac{1}{10}(16, 16, 16, 16, 16, 4, 4, 4, 4, 16, 4, 4)^t,$$

the elements of $\mathcal{S} = \{x \in \mathbb{R}^{12} \mid H(f)x = 2b\}$ have the form[2]

$$\left(t + \frac{3}{10}, \frac{3}{10} + s, -s - t + \frac{1}{10}, -s - t + \frac{1}{10}, \frac{3}{10} + s, s, -s - t - \frac{1}{5}, -s - t - \frac{1}{5}, s, t + \frac{3}{10}, t, t\right)$$

with $(s, t) \in \mathbb{R}^2$. But it is impossible to find $t \geq 0$, $s \geq 0$ with $-s - t - \frac{1}{5} \geq 0$. This shows that $\mathcal{S}$ does not contain non-negative elements and $B \notin \Delta_4$.

[1] Calculated with Maple.
[2] Calculated with Maple.

Example 4.4. Choose again the matrix B from Example 4.3. With the numbering $\pi_1 = (123)$, $\pi_2 = (132)$, $\pi_3 = (124)$, $\pi_4 = (142)$, $\pi_5 = (134)$, $\pi_6 = (143)$, $\pi_7 = (234)$, $\pi_8 = (243)$, $\pi_9 = (12)(34)$, $\pi_{10} = (13)(24)$, $\pi_{11} = (14)(23)$ the matrix $H(g)$ reads

$$H(g) = 2\begin{pmatrix} 6 & 3 & 3 & 2 & 2 & 3 & 3 & 2 & 4 & 4 & 4 \\ 3 & 6 & 2 & 3 & 3 & 2 & 2 & 3 & 4 & 4 & 4 \\ 3 & 2 & 6 & 3 & 3 & 2 & 2 & 3 & 4 & 4 & 4 \\ 2 & 3 & 3 & 6 & 2 & 3 & 3 & 2 & 4 & 4 & 4 \\ 2 & 3 & 3 & 2 & 6 & 3 & 3 & 2 & 4 & 4 & 4 \\ 3 & 2 & 2 & 3 & 3 & 6 & 2 & 3 & 4 & 4 & 4 \\ 3 & 2 & 2 & 3 & 3 & 2 & 6 & 3 & 4 & 4 & 4 \\ 2 & 3 & 3 & 2 & 2 & 3 & 3 & 6 & 4 & 4 & 4 \\ 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 8 & 4 & 4 \\ 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 8 & 4 \\ 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 8 \end{pmatrix}.$$

The rank of $H(g)$ amounts to 9, and $\mathcal{R}$ is an affine subspace of $\mathbb{R}^{11}$ of dimension 2, while

$$c = \left(3, 3, 3, 3, \frac{18}{10}, \frac{18}{10}, \frac{18}{10}, \frac{18}{10}, 4, \frac{28}{10}, \frac{28}{10}\right)^t.$$

The elements of $\mathcal{R} = \{x \in \mathbb{R}^{11} \mid H(g)x = 2c\}$ have the form[3]

$$\left(\frac{3}{10} + t, \frac{1}{10} - t - s, \frac{1}{10} - t - s, \frac{3}{10} + t, t, -\frac{1}{5} - t - s, -\frac{1}{5} - t - s, t, \frac{3}{10} + s, s, s\right)$$

with $(s, t) \in \mathbb{R}^2$. As above, $\mathcal{R} \cap \mathbb{F}^p = \emptyset$ and $B \notin \Delta_4$. Moreover, for $s = 0$ and $t = -\frac{3}{10}$

$$y = \left(0, \frac{2}{5}, \frac{2}{5}, 0, \frac{-3}{10}, \frac{1}{10}, \frac{1}{10}, \frac{-3}{10}, \frac{3}{10}, 0, 0\right) \quad \text{and} \quad g(y) = 0,$$

showing that B belongs to the linear span of A_4:

$$B = \frac{2}{5}(123) + \frac{2}{5}(124) + \frac{-3}{10}(134) + \frac{1}{10}(145) + \frac{1}{10}(234) + \frac{-3}{10}(243) + \frac{3}{10}(12)(34) + \frac{3}{10}1_{S_n}.$$

Example 4.5. Omitting the condition $\pi_m \prec A$ leads nevertheless to linear even representations. Choose

$$A = \frac{1}{n+1}ee^t + (12)$$

that is an even doubly-stochastic matrix for $n = 5$, since

$$A = \frac{1}{6}\left((123) + (132) + (12)(45) + (14)(35) + (15)(42) + (25)(34)\right).$$

But we also have

$$A = \frac{1}{6}\left(1_{S_n} + (12345) + (13524) + (14253) + (15432) + (12)\right)$$

yielding that the transposition (12) belongs to the linear span of A_5,

$$\begin{aligned}(12) = {} & (123) + (132) + (12)(45) + (14)(35) + (15)(42) + (25)(34) \\ & - 1_{S_n} - (12345) - (13524) - (14253) - (15432)\end{aligned}$$

[3] Calculated with Maple.

and

$$\min_{x\in\mathbb{R}^{\frac{n!}{2}}} \left\| (12) - \sum_{\pi\in A_n} x_\pi \pi \right\| = 0.$$

Example 4.6. The control of the positivity of the coefficient vector y is indispensable. The matrix

$$A = \frac{1}{5}\,(2(123) + 2(134) + 2(145) - (12345)) = \frac{1}{5}\begin{pmatrix} 0 & 1 & 2 & 2 & 0 \\ 0 & 4 & 1 & 0 & 0 \\ 2 & 0 & 2 & 1 & 0 \\ 2 & 0 & 0 & 2 & 1 \\ 1 & 0 & 0 & 0 & 4 \end{pmatrix}$$

lies in the linear span of A_5 with $y^t\mathbf{e}_4 = 1$ and $f(y) = 0$. But A is not even, because for $\pi = 1_{S_n}$ and for $k = 1$, Mirsky's condition (1) is not satisfied.

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